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STABILITY IN THE SHEAR LAYER OF A COMPRESSIBLE GAS

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In the mixing of two parallel streams of a viscous gas, moving at different velocities, near the boundary of separation a flow is formed that is referred to as an ordinary free shear layer. Such flows in actual practice are encountered in the boundary layer of a jet discharging into a submerged space, in the wake trailing a nonsymmetrically streamlined body, etc. The free shear flows are extremely unstable to small perturbations, i.e., the shear layer of an incompressible gas, for example, is unstable for all Reynolds numbers Re [1]. The stability of the compressible shear layer in the case of finite Re has, apparently, not been studied earlier. Without provision for viscosity, this problem is solved in [2-4], with a number of additional simplifications having been introduced: the temperature throughout the entire flow was assumed to be constant and the dynamic profile was given by the function $U(y) = \tanh y$.

In the present study in investigating the stability of the compressible shear layer we assume the gas to be viscous and capable of conducting heat, with the velocity and temperature profiles calculated from corresponding boundary-layer equations [5]. Approximations of an incompressible or inviscid gas are thereby attained in the form of limit cases in which the Mach number $M \to 0$ or $Re \to \infty$. The calculations were performed numerically by the orthogonalization method [6]. It is demonstrated that when $M \lesssim 1$ the stability of the flow is determined by wave perturbations exhibiting a phase velocity $c_r = 0$ and a zero critical Reynolds number Re_*. With an increase in M the region of unstable wave numbers narrows. When $M \gtrsim 1$, as in the case of the inviscid problem [3], stability is determined by traveling waves with $c_r \neq 0$ (the second perturbation mode). It has been observed that for the second mode Re_* is different from zero and diminishes as M increases. We have constructed the neutral stability curves, the eigenfunctions, and we have studied the relationship between the characteristics of stability and M for the case in which $0 \leq M \leq 2$.

1. Let us examine the plane flow in the shear layer of a compressible viscous heat-conducting gas. We will assume the gas to be ideal, with constant heat capacities c_V and $c_p = \gamma c_V$, viscosity μ, and thermal conductivity k directly proportional to temperature, so that the Prandtl number $Pr = \mu c_p/k$ is constant and that the second viscosity is equal to zero. The Navier–Stokes equations, written in dimensionless form, in this case have the form

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho u_i)}{\partial x_i} = 0, \quad \rho\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right) + \frac{\partial p}{\partial x_i} = \frac{1}{\mathrm{Re}}\frac{\partial \sigma_{ij}}{\partial x_j},$$
$$\rho\left(\frac{\partial\theta}{\partial t} + u_j\frac{\partial\theta}{\partial x_j}\right) + (\gamma-1)\rho\theta\frac{\partial u_j}{\partial x_j} = \frac{\gamma}{\mathrm{Re\,Pr}}\frac{\partial}{\partial x_j}\left(\mu\frac{\partial\theta}{\partial x_j}\right) + \frac{\gamma(\gamma-1)\mathrm{M}^2}{\mathrm{Re}}\sigma_{ij}e_{ij}, \tag{1.1}$$
$$\mu(\theta) = \theta, \quad p = \rho\theta/\gamma\mathrm{M}^2,$$
$$\sigma_{ij} = 2\mu e_{ij} - \frac{2}{3}\mu e_{kk}\delta_{ij}, \quad e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i, j, k = 1, 2,$$
$$\mathrm{M} = U_*/\sqrt{\gamma R T_*}, \quad \mathrm{Re} = \rho_* U_*\delta/\mu_*, \quad \delta = (\pi\mu X/\rho_* U_*)^{1/2}.$$

Here $u_1 \equiv u$ and $u_2 \equiv v$ are the longitudinal and transverse components of velocity in the direction of the $x_1 \equiv x$ and $x_2 \equiv y$ axes, respectively; ρ, p, and θ are the density, pressure, and temperature of the gas; R is the gas constant. The region in which the independent variables x and y change represents the entire plane $-\infty < x < \infty$, $-\infty < y < \infty$. We have

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taken the thickness of the boundary layer δ between the flows as our scale of length and X represents the dimensional longitudinal coordinate. Velocity, density, and temperature have been referred to their values in the uniform unperturbed flow as $y \to \infty$ [identified in (1.1) with asterisks], so that $u|_{y\to\infty} = 1$, $\theta|_{y\to\infty} = 1$. The constant values for velocity and temperature in the other flow have been denoted as $u|_{y\to-\infty} = m$, $\theta|_{y\to-\infty} = \kappa$.

To study the stability of the shear layer we will, as is usual practice, seek a solution in the form of a superposition of the main laminar flow taken for a fixed value of the coordinate $x = x_0$ (the quasiparallel approximation) and the perturbations periodic with respect to x, i.e., traveling low-amplitude waves:

$$\rho = \rho_0(x_0, y) + \tilde{\rho}(x, y, t),\ u = U(x_0, y) + \tilde{u}(x, y, t), \tag{1.2}$$
$$v = \tilde{v}(x, y, t),\ p = \Pi(x_0, y) + \tilde{p}(x, y, t),\ \theta = T(x_0, y) + \tilde{\theta}(x, y, t);$$
$$\{\tilde{\rho}, \tilde{u}, \tilde{v}, \tilde{p}, \tilde{\theta}\} = \{\boldsymbol{\rho}(y), \mathbf{u}(y), \mathbf{v}(y), \mathbf{p}(y), \boldsymbol{\theta}(y)\}e^{i\alpha(x-ct)},\quad c = c_r + ic_i \tag{1.3}$$

(α is the wave number of the periodic perturbation, with the period $2\pi/\alpha$; c_r is the phase velocity of the wave). Calculation of the main-flow parameters is accomplished on the basis of boundary-layer theory [5], the pressure $\Pi = 1/\gamma M^2$ is constant, and it follows from (1.1) that $\rho_0 = 1/T$ (the calculation of the main flow and the relationship of x_0 to Re can be found in the Appendix).

Having substituted (1.2) and (1.3) into (1.1) and leaving only the terms of the first order of smallness with respect to perturbation amplitude in the equations, we arrive at the following system of ordinary differential equations:

$$D\boldsymbol{\rho} + \frac{\eta}{T} - \frac{1}{T^2}\frac{dT}{dy}\mathbf{v} = 0, \tag{1.4}$$
$$D\mathbf{u} + \frac{dU}{dy}\mathbf{v} + i\alpha T\mathbf{p} = \frac{T}{\mathrm{Re}}\left[\mu_0\Delta\mathbf{u} + \frac{i\alpha\mu_0\eta}{3} + \frac{dT}{dy}\left(\frac{d\mathbf{u}}{dy} + i\alpha\mathbf{v}\right) + \frac{d}{dy}\left(\frac{dU}{dy}\boldsymbol{\theta}\right)\right],$$
$$D\mathbf{v} + T\frac{d\mathbf{p}}{dy} = \frac{T}{\mathrm{Re}}\left[\mu_0\Delta\mathbf{v} + \frac{\mu_0}{3}\frac{d\eta}{dy} + \frac{2}{3}\frac{dT}{dy}\left(2\frac{d\mathbf{v}}{dy} - i\alpha\mathbf{u}\right) + i\alpha\frac{dU}{dy}\boldsymbol{\theta}\right],$$
$$D\boldsymbol{\theta} + \frac{dT}{dy}\mathbf{v} + (\gamma - 1)T\eta = \frac{\gamma T}{\mathrm{Re\,Pr}}\left(\mu_0\Delta\boldsymbol{\theta} + \frac{d^2T}{dy^2}\boldsymbol{\theta} + 2\frac{dT}{dy}\frac{d\boldsymbol{\theta}}{dy}\right) +$$
$$+ \frac{\gamma(\gamma-1)M^2}{\mathrm{Re}}T\left[2\mu_0\frac{dU}{dy}\left(\frac{d\mathbf{u}}{dy} + i\alpha\mathbf{v}\right) + \left(\frac{dU}{dy}\right)^2\boldsymbol{\theta}\right],$$
$$\mathbf{p} = (\boldsymbol{\theta}/T + T\boldsymbol{\rho})/\gamma M^2,\ \mu_0 = \mu(T),\ \eta = i\alpha\mathbf{u} + d\mathbf{v}/dy,$$
$$D = i\alpha(U - c),\ \Delta = d^2/dy^2 - \alpha^2.$$

Equations (1.4), enhanced by the requirement of the limitations imposed on the perturbations at infinity

$$|\boldsymbol{\rho}|, |\mathbf{u}|, |\mathbf{v}|, |\mathbf{p}|, |\boldsymbol{\theta}| < \infty \text{ as } |y| \to \infty, \tag{1.5}$$

make up the eigenvalue problem for c. The flow is stable if $c_i < 0$, and it is unstable when $c_i > 0$. With $c_i = 0$ the flow is neutrally stable.

Let us note that at the inviscid limit as $\mathrm{Re} \to \infty$ system (1.4) reduces to a single equation for the pressure amplitude [7]

$$\frac{d^2\mathbf{p}}{dy^2} - \left(\frac{2}{U-c}\frac{dU}{dy} - \frac{1}{T}\frac{dT}{dy}\right)\frac{d\mathbf{p}}{dy} + \alpha^2\left[\frac{M^2(U-c)^2}{T} - 1\right]\mathbf{p} = 0, \tag{1.6}$$

whose asymptotic solution as $|y| \to \infty$ is written as

$$\mathbf{p} \sim \exp\left(\pm\alpha\sqrt{1 - M^2(U-c)^2/T}\,y\right). \tag{1.7}$$

For example, let $y \to -\infty$. Then from (1.7) we see that if the following condition has been satisfied, and namely:

$$c_i = 0,\ |m - c_r| > \sqrt{\varkappa}/M, \tag{1.8}$$

the perturbation oscillates, without attenuation as $y \to -\infty$. Analogously, when $y \to +\infty$ the solution of the inviscid equation is not attenuated, provided that

$$c_i = 0,\ |1 - c_r| > 1/\mathrm{M}. \tag{1.9}$$

In dimensionless variables 1/M is the speed of sound in the uniform flow as $y \to \infty$, and it is $\sqrt{\varkappa}/\mathrm{M}$ as $y \to -\infty$, so that conditions (1.8) or (1.9) therefore characterize the perturbations propagated at supersonic speeds relative to the "upper" (or "lower") flow of gas. As is demonstrated in A.3, such "supersonic" perturbations for $\mathrm{M} \gtrsim 1$ play a decisive role in the complete stability problem (1.4), (1.5).

2. For the numerical solution of Eqs. (1.4) it is convenient to rewrite these in the form of a system of six first-order equations [8] and to take the Dorodnitsyn variables φ as the new independent variable:

$$d\mathbf{f}(\varphi)/d\varphi = G(\varphi)\mathbf{f},\ \mathbf{f} = (\mathbf{u}, \mathbf{u}', \mathbf{v}, \mathbf{p}, \boldsymbol{\theta}, \boldsymbol{\theta}'). \tag{2.1}$$

The prime here and below denotes derivatives with respect to φ, the relationship between the variables φ and y, and the dependence of the main-flow velocity and temperature U, T as functions of φ are given by formulas (A.6)-(A.9). In Eq. (2.1) G is a matrix with dimensions of 6 × 6, whose elements differ from zero are presented as follows:

$$\begin{gathered}
G_{12} = 1,\ G_{21} = \mathrm{Re}D + \alpha^2T^2,\ G_{23} = \mathrm{Re}U'/T - 4i\alpha T'/3,\\
G_{24} = i\alpha T(\mathrm{Re} + \gamma \mathrm{M}^2TD/3),\ G_{25} = -i\alpha TD/3 - (U'/T)',\\
G_{26} = -U'/T,\ G_{31} = -i\alpha T,\ \ G_{33} = T'/T,\ G_{34} = -\gamma\mathrm{M}^2TD,\\
G_{35} = D,\ G_{41} = -10i\alpha T'/3Q,\ G_{42} = -i\alpha T/Q,\ G_{43} =\\
= (4T''/3T - \mathrm{Re}D - \alpha^2T^2)/Q,\ G_{44} = -4\gamma\mathrm{M}^2(2T'D + i\alpha U'T)/3Q,\\
G_{45} = (4DT'/T + 7i\alpha U')/3Q,\ G_{46} = 4D/3Q,\ Q = \mathrm{Re} + 4\gamma\mathrm{M}^2TD/3,\\
G_{56} = 1,\ G_{62} = -2(\gamma - 1)\mathrm{M}^2\mathrm{Pr}U',\ G_{63} = \mathrm{RePr}T'/T - 2i\alpha(\gamma -\\
-1)\mathrm{M}^2\mathrm{Pr}U'T,\ G_{64} = -(\gamma - 1)\mathrm{M}^2\mathrm{RePr}TD,\\
G_{65} = \mathrm{RePr}D + \alpha^2T^2 - (T'/T)' - (\gamma - 1)\mathrm{M}^2\mathrm{Pr}U'^2/T,\\
G_{66} = -T'/T.
\end{gathered} \tag{2.2}$$

The procedure of the numerical solution involves determination of the fundamental system of solutions (2.1) in the external region $|\varphi| \to \infty$, the extension of the eigenvectors by numerical integration to the "inside" of the shear layer and the subsequent combining of the "upper" and "lower" general solutions at the point $\varphi = 0$.

As $|\varphi| \to \infty$ the coefficients from (2.2) are constant and a solution proportional to $e^{\lambda\phi}$ may be sought. From (2.1) we then find the following system of algebraic equations:

$$\begin{gathered}
(\mathrm{Re}D + \alpha^2T^2)\mathbf{u} + i\alpha T(\mathrm{Re} + \gamma\mathrm{M}^2TD/3)\mathbf{p} - i\alpha D\mathrm{T}\boldsymbol{\theta}/3 = \lambda^2\mathbf{u},\\
-i\alpha T\mathbf{u} - \gamma\mathrm{M}^2TD\mathbf{p} + D\boldsymbol{\theta} = \lambda\mathbf{v},\\
-i\alpha T\lambda\mathbf{u} - (\mathrm{Re}D + \alpha^2T^2)\mathbf{v} + 4D\lambda\boldsymbol{\theta}/3 = (\mathrm{Re} + 4\gamma\mathrm{M}^2TD/3)\lambda p,\\
-(\gamma - 1)\mathrm{M}^2\mathrm{RePr}TD\mathbf{p} + (\mathrm{RePr}D + \alpha^2T^2)\boldsymbol{\theta} = \lambda^2\boldsymbol{\theta}.
\end{gathered} \tag{2.3}$$

We will show that the roots λ of the characteristic equation for system (2.3) (i.e., the eigenvalues of the matrix $G|_{\varphi|\to\infty}$) can be found analytically. Multiplying the first equation by 1/T, combining it with the second, multiplied by $-i\alpha$, and eliminating $\mathbf{p}$ and $\boldsymbol{\theta}$ with the aid of the third of the equations in (2.3), we obtain an equation for the rotation $\boldsymbol{\omega}$. It has the same form as in an incompressible gas:

$$(\mathrm{Re}D + \alpha^2T^2)\boldsymbol{\omega} = \lambda^2\boldsymbol{\omega},\ \omega = (1/T)\lambda\mathbf{u} - i\alpha\mathbf{v}. \tag{2.4}$$

From this we immediately determine one of the values of λ_1^2. Then, having eliminated $\mathbf{u}$ and $\mathbf{v}$ from the third and fourth equations in (2.3), we come to a uniform system of two related equations for $\mathbf{p}$ and $\boldsymbol{\theta}$. Equating the determinant to zero, it is easy to prove that as a result we will have an equation that is quadratic relative to λ^2, from which we find the remaining roots λ_2^2 or λ_3^2. Thus,

$$\lambda_i^2 = \alpha^2T^2 + \Delta_i,\ i = 1, 2, 3, \tag{2.5}$$

$$\Delta_1 = \mathrm{Re}\,D,\quad \Delta_{2,3} = \frac{1}{2}\mathrm{Pr}\,\Delta_1\frac{1 + q\left(\frac{4}{3} + \frac{\gamma}{\mathrm{Pr}}\right)}{1 + \frac{4}{3}q\gamma}\times$$

$$\times\left\{1\pm\sqrt{1-\frac{4q}{\Pr}\frac{1+\frac{4}{3}q\gamma}{\left[1+q\left(\frac{4}{3}+\frac{\gamma}{\Pr}\right)\right]^2}}\right\},\quad q=\frac{M^2TD}{\mathrm{Re}}.$$

The corresponding eigenvectors $\mathbf{w}_i$ of the matrix $G|_{\varphi\to\infty}$ are obtained after substitution of (2.5) into (2.1) and we will write their components w_i^j as follows:

$$w_i^1=1,\ w_i^2=\lambda_i,\ w_i^6=\lambda_i w_i^5,\ i=1,2,3,$$
$$w_1^3=-i\alpha T/\lambda_1,\ w_1^4=w_1^5=w_1^6=0,$$
$$w_k^3=-\frac{i\alpha T}{\lambda_k}+\frac{1}{\lambda_k}\left[D-\frac{\gamma}{(\gamma-1)\,\mathrm{Re}\Pr}\left(\mathrm{Re}\Pr D+\alpha^2T^2-\lambda_k^2\right)\right]w_k^5,$$
$$w_k^4=\frac{\mathrm{Re}\Pr D+\alpha^2T^2-\lambda_k^2}{(\gamma-1)\,M^2\,\mathrm{Re}\Pr TD}w_k^5,$$
$$w_k^5=\frac{\mathrm{Re}\,D+\alpha^2T^2-\lambda_k^2}{i\alpha\left[\frac{TD}{3}-\frac{(\mathrm{Re}+\gamma M^2TD/3)\left(\mathrm{Re}\Pr D+\alpha^2T^2-\lambda_k^2\right)}{(\gamma-1)\,M^2\,\mathrm{Re}\Pr D}\right]},\quad k=2,\ 3. \tag{2.6}$$

In the limit case of large Re formulas (2.5) take on a simpler appearance:

$$\lambda_1^2=\alpha^2T^2+\mathrm{Re}\,D,\ \lambda_2^2=\alpha^2T^2+\mathrm{Re}\Pr D,\ \lambda_3^2=\alpha^2T^2+D^2M^2T. \tag{2.7}$$

In the external region the particular solutions of the form $\mathbf{w}_ie^{\lambda_i\varphi}$ exhibit the clear physical sense of waves of rotation, temperature (or entropy), and of pressure, i.e., three types of elementary excitations in a compressible heat-conducting gas [9]. Let us note that the solution for λ_3 (the pressure wave) as $\mathrm{Re}\to\infty$ changes into the solution of the inviscid problem [compare (1.7) with (2.7)]. The overall solution of (2.1) as $\varphi\to\infty$, satisfying (1.5), is written as a linear combination of the three particular solutions:

$$\mathbf{f}|_{\varphi\to\infty}=\sum_{i=1}^{3}C_i\mathbf{w}_ie^{\lambda_i\varphi}, \tag{2.8}$$

where for λ_i we should take the value of the quadratic root from λ_i^2, for which $\mathrm{Real}(\lambda_i)<0$ [in (2.5) and (2.6) in this case we have $U = 1$, $T = 1$]. The case in which for one of the λ_i $\mathrm{Real}(\lambda_i) = 0$ corresponds to the presence of a continuous spectrum in problem (1.4), (1.5), which is not dealt with in this study. As $\varphi\to-\infty$ we must assume in (2.5) and (2.6) that $U = m$, $T = \kappa$ and we have to choose the sign of the quadratic roots so that $\mathrm{Real}(\lambda_i)>0$. We will denote the eigenvalues and eigenvectors of the matrix $G|_{\varphi\to-\infty}$ in terms of $\lambda_4,\lambda_5,\lambda_6,\mathbf{w}_4,\mathbf{w}_5,\mathbf{w}_6$.

In the numerical solution system (2.1) is integrated with the initial data $\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3$ from $\varphi=\infty$ to $\varphi=0$, and then with $\mathbf{w}_4,\mathbf{w}_5,\mathbf{w}_6$ from $\varphi=-\infty$, also to $\varphi=0$. The vectors obtained after integration are denoted $\mathbf{z}_1,\ldots,\mathbf{z}_6$. With $\varphi=0$ the eigenfunction of the problem must be a linear combination of both $\mathbf{z}_1,\mathbf{z}_2,\mathbf{z}_3$, and $\mathbf{z}_4,\mathbf{z}_5,\mathbf{z}_6$:

$$C_1\mathbf{z}_1+C_2\mathbf{z}_2+C_3\mathbf{z}_3=C_4\mathbf{z}_4+C_5\mathbf{z}_5+C_6\mathbf{z}_6. \tag{2.9}$$

The condition for the existence of a nontrivial solution for (2.9) is the vanishing of the determinant $F=|z_i^j|$:

$$F(c,\alpha,M,\mathrm{Re})=0. \tag{2.10}$$

The fourth order-of-accuracy Runge–Kutta scheme was used in the integration of (2.1). To avoid the rapid accumulation of errors associated with the presence of the small parameter in (1.4) for the higher derivatives, we used the orthogonalization method from [6]. Equation (2.10) was solved with the Newton iteration method in the determination of c.

It should be underscored that in the actual conduct of the calculations the region of convergence for the iteration method may be extremely narrow, since in the case of large Re the solution for (2.1) and (2.8) is close to that for the inviscid problem (1.6), (1.7) and the vectors $\mathbf{z}_3,\mathbf{z}_6$ are virtually linearly dependent. Therefore, in the present study

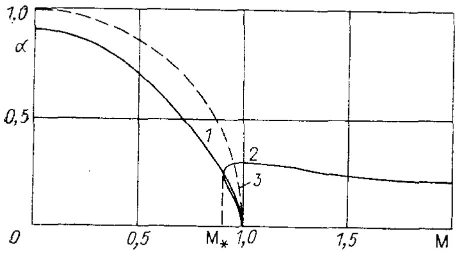

Fig. 1

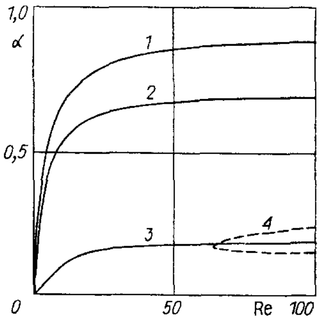

Fig. 2

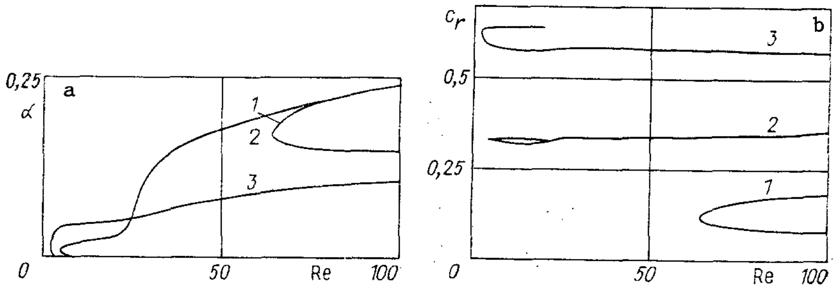

Fig. 3

we carried out the calculations in the following manner. We assumed the constant C_6 to be equal to unity, and one of the equations (usually the equation for p) was separated from system (2.9). The system of five equations, whose matrix no longer contains any rows or columns that are nearly linear functions, was then solved for $C_1, \ldots, C_5$. The found values for the constants were substituted into the remaining equation and instead of condition (2.10) it became necessary to satisfy this equation. Boundary conditions of the form of (2.8) were imposed with sufficiently large $|\varphi| = L$ in the main portion of the calculations with L = 5.5. The number of points on the interval (−L, L) was usually N = 129, while in the case of large Re the calculations were sometimes carried out with N = 257. As a rule, the iteration process was brought to a conclusion when the change in the unknown quantity in the next iteration step proved to be smaller than 10^{-6}.

3. The numerical calculations in this study have been carried out for the case in which the mixing flows have identical temperature ($\kappa = 1$) and move with equal oppositely directed velocities (m = −1). Setting m = −1 does not reduce the generality of our examination, since it simply corresponds to a transition to a reckoning system moving at a velocity equal to the half-sum of the flow velocities. In all of the calculations Pr = 0.72, $\gamma = 1.4$. The results of the stability calculations can be seen in Figs. 1-6.

Figure 1, in the coordinates α, M, shows the neutral stability curves ($c_i = 0$) for $Re = 10^3$, i.e., virtually under the conditions of the inviscid problem. Neutral curves 1 and 2 correspond to the two different branches of the dispersion equation (2.10), i.e., to the various perturbation "modes." However, for purposes of comparison neutral curve 3 with the first mode $\alpha^2 + M^2 = 1$ is given here, and it was derived analytically in [2]. We can see that when $M \le 1$ the selection in the inviscid problem of simpler distributions of velocity and temperature for the main flow having a form such as U(y) = tanh y, T(y) = 1 qualitatively does not change the behavior of the neutral curve. In the absence of compressibility (M = 0) the perturbations are unstable when $\alpha < \alpha_* = 0.915$ ($\alpha < 1$ in [2]). On the whole, compressibility with $M \neq 0$ exerts a stabilizing effect, i.e., the region of unstable wave numbers (bounded by the axis α = 0 and the neutral curves) becomes constricted as M increases, and with M = 1 total stabilization of the first mode sets in. A characteristic property of the first mode, associated with the symmetry of the main flow when m = −1, $\kappa = 1$ (see Appendix), is the fact that the phase velocity is equal to zero ($c_r = 0$) for all α, Re, M, and c_i, i.e., the first mode is a standing wave. Moreover, if c_i also vanishes, then in the shear layer we find a "replacement of stability," i.e., a new steady state sets in.

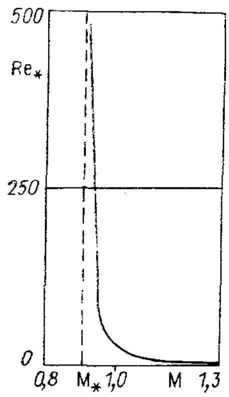

Fig. 4

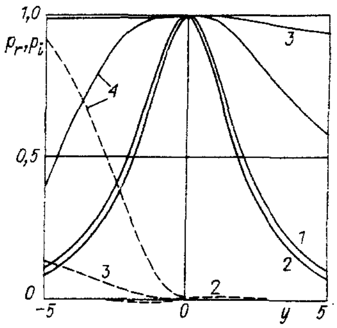

Fig. 5

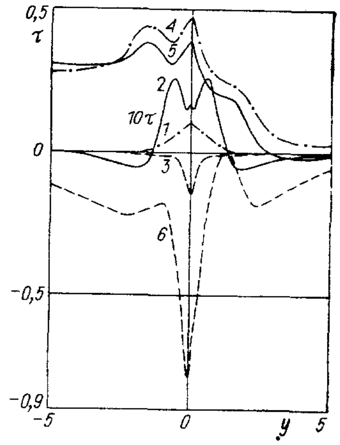

Fig. 6

For the second mode (curve 2) the calculations show that $c_r \neq 0$ also changes in accordance with condition (1.8) in the interval $1 - 1/M < c_r < 1$, so that a wave of this type is propagated at subsonic speed relative to the "upper" flow and with supersonic speed relative to the "lower" flow. At first glance, the existence of such perturbations contradicts the properties of symmetry for the main flow. The paradox is resolved, however, by the circumstances that if $c_r + ic_i$, i.e., the eigenvalue of problem (1.4), (1.5) with the eigenfunctions $\rho(y)$, $u(y)$, $v(y)$, $p(y)$, $\theta(y)$, then given the same α, M, Re, and $-c_r + ic_i$, i.e., the eigenvalue with the eigenfunctions $\rho^*(-y)$, $-u^*(-y)$, $-v^*(-y)$, $p^*(-y)$, $\theta^*(-y)$ (the asterisk indicates complex-conjugate quantities). Indeed, such a formulation changes Eq. (1.4) into one that is complex-conjugate. This means that the second mode is actually two modes exhibiting coincident neutral curves. The minimum Mach number at which neutral oscillations arise with $c_r \neq 0$, $M_* = 0.906$ (see Fig. 1), so that in the comparatively narrow range $M_* < M < 1$ there exist three unstable oscillating modes simultaneously. One of these is a standing wave, and the two other modes, exhibiting identical growth coefficients for αc_i, travel in opposite directions at different velocities. In the following, when dealing with the second mode, we will have in mind that wave which propagates in the positive direction of the x axis. The results from the calculation for the second traveling wave are analogous but have the obvious changes.

Figure 2 shows the neutral curves for the first mode in the plane α, Re for various M: curves 1-3) M = 0 (the incompressible shear layer); 0.5; 0.95. To make the picture more complete, with M = 0.95 here we have also a plot of the second-mode neutral curve (line 4). We see that with an increase in M (curves 2 and 3) the Re_* for the first mode remains equal to zero, i.e., the shear layer is unstable when M < 1 for any Re. The effect of compressibility on the first mode thus reduces to a reduction of the region of unstable wave numbers.

Figure 3 shows the results from the calculation of the stability in the second mode. Figure 3a shows neutral curves 1-3 for M = 0.95, 1.2, 2.0. Unlike the first mode $Re_* \neq 0$, it rapidly diminishes with an increase in M. The stabilizing effect of viscosity becomes noticeable to a greater degree when M = 0.95 (Re_* for curve 1 exceeds Re_* of curves 2 and 3 by more than an order of magnitude). Figure 3b shows the c_r of the neutral perturbations as a function of Re for the same three values of M. The quantity c_r changes quite insignificantly when M = 1.2 or M = 2.0 (see curves 2 and 3 in Fig. 3b, for the first of these c_r at the upper and lower branches of the neutral curve virtually coincide).

Figure 4 shows Re_* as a function of M. Comparison of this figure with Figs. 1 and 3a makes it possible to describe the qualitative behavior of the neutral second-mode curve with a change in M as follows. With $M \geq 1$ the asymptote of its lower branch is represented by the axis α = 0, $Re_* \sim 1$. When $M \to M_*$, Re_* rapidly increases, the neutral curve shifts into the region of large Re, and the lower asymptote approaches the upper (see Fig. 1). In this range of M the effect of viscosity on the oscillations of the second mode is at its maximum. If we also take into consideration the presence of the first mode, we can draw the conclusion that the shear layer is most stable when $M \approx 1$. With M = 1 the first mode is completely stabilized and for the oscillations of the second mode we have $Re_* = 32.8$.

Figure 5 shows the behavior of the eigenfunctions for pressure perturbations $\mathbf{p}(y)$ [these have been normalized so that $\mathbf{p}(0) = 1$] of the neutral oscillations of the first (M = 0.5) and second (M = 1.2) modes for two values of Re. Curves 1 and 2 pertain to the first mode and correspond to Re = 20, α = 0.620 and Re = 100, α = 0.697. For the second mode Re = 20, α = 0.028, c_r = 0.330 (curve 3), Re = 100, α = 0.230, c_r = 0.351 (curve 4). The solid lines represent the real part of the eigenfunctions, while the dashed lines represent the imaginary parts [the imaginary part of $\mathbf{p}(y)$ in case 2 is very small and is not shown in Fig. 5]. We can see that the behavior of the eigenfunctions for the two modes differ completely, in particular, the eigenfunctions of the second mode virtually do not diminish as $y \to -\infty$, i.e., the perturbations radiate into the external flow. It is also clearly evident that if curves 1 and 2, corresponding to various Re, differ weakly from one another, then 3 and 4 will differ significantly. The pressure perturbation for the case in which Re = 20 (curve 3) undergoes virtually no change across the shear layer. This is associated with the fact that this is a long-wave perturbation (α = 0.028), with a wavelength of $2\pi/\alpha$, significantly exceeding the thickness of the shear layer. The influence of the viscosity on the second mode thus is considerably stronger than its effect on the first mode.

Figure 6 shows the Reynolds stress waves $\tau(y)$ averaged over the length

$$\tau = -\rho_0 \overline{\text{Real}\ \tilde{u}\ \text{Real}\ \tilde{v}} = -(1/4)(\mathbf{u}\mathbf{v}^* + \mathbf{u}^*\mathbf{v})/T. \tag{3.1}$$

They characterize the exchange of energy between the perturbations and the main flow. Curves 1-3 correspond to α = 0.6, 0.697, 0.8 when M = 0.5 (the first mode); curves 4-6 correspond to α = 0.15, 0.230, 0.3 when M = 1.2 (the second mode), the solid lines show τ for the neutral oscillations while the dashed lines represent the attenuating oscillations; the dashed-dotted lines represent the increasing oscillations (Re = 100). The eigenvalues of c for curves 1-6 are as follows: c = 0.0965i; 0; −0.107i; 0.310 + 0.0443i; 0.330; 0.342 − 0.277i. Curve 2 has been plotted on another scale ($\tau \times 10$). Let us note that in the case of inviscid neutral perturbations of the first mode the Reynolds stresses are identically equal to zero, while for the second mode in this case τ experiences a discontinuity at the point y = 0, remaining constant on both sides of that point [3].

APPENDIX

The profiles of velocity and temperature for the steady-state main flow in the shear layer, satisfying the equations of the boundary layer with the boundary conditions

$$U|_{y\to\infty} = 1,\ T|_{y\to\infty} = 1,\ U|_{y\to-\infty} = m,\ T|_{y\to-\infty} = \varkappa, \tag{A.1}$$

have the following form [5]:

$$U(\psi) = 1 + (1/2)(m-1)(1 - \operatorname{erf}\psi); \tag{A.2}$$

$$T(\psi) = 1 + \frac{1}{2}(\varkappa - 1)\left[1 - \operatorname{erf}(\psi\sqrt{\Pr})\right]$$

$$+ (\gamma - 1)M^2 \int_{\psi}^{\infty} \left[\frac{dU(z)}{dz}\right]^{\Pr} \left(\int_0^z \left[\frac{dU(t)}{dt}\right]^{2-\Pr} dt\right) dz, \tag{A.3}$$

where ψ is determined by the formula

$$\frac{y}{2\sqrt{x}} = \frac{1}{\sqrt{\mathrm{Re}}} \int_0^{\psi} T(z)\, dz. \tag{A.4}$$

Here the variables x and y have been referred to the thickness δ of the shear layer [see (1.1)]. The Reynolds number, constructed on the basis of δ, is associated in this case with the coordinate x by the formula

$$\mathrm{Re} = \pi x. \tag{A.5}$$

The main flow in the stability problem is treated in the fixed cross section $x = x_0$, where according to (A.5), $x_0 = \mathrm{Re}/\pi$. Then in the place of (A.4) we will write

$$y = \frac{2}{\sqrt{\pi}} \int_0^{\psi} T(z)\, dz.$$

It is more convenient to use the variable $\varphi = 2\psi/\sqrt{\pi}$, so that

$$y = \int_0^{\varphi} T\,dz. \quad (A.6)$$

Having substituted U from (A.2) into (A.3), with consideration of (A.6), we have the final formulas for U and T:

$$U(\varphi) = 1 + (1/2)(m - 1)[1 - \mathrm{erf}\,(\sqrt{\pi}\varphi/2)]; \quad (A.7)$$

$$T(\varphi) = 1 + \frac{1}{2}(\varkappa - 1)\left[1 - \mathrm{erf}\left(\sqrt{\pi\,\mathrm{Pr}}\,\varphi/2\right)\right] + \frac{\mathrm{Pr}\,(\gamma - 1)\,\mathrm{M}^2\,(m - 1)^2}{4\sqrt{2 - \mathrm{Pr}}} \int_{\varphi}^{\infty} \Phi(z)\,dz; \quad (A.8)$$

$$\Phi(z) = \exp\,(-\pi \mathrm{Pr} z^2/4)\,\mathrm{erf}\,(\sqrt{\pi(2 - \mathrm{Pr})}z/2). \quad (A.9)$$

If we now assume that $m = -1$, $\kappa = 1$, then U will be an antisymmetric, and T will be a symmetric, functions of the variable φ and, consequently, according to (A.6), also of the variable y.

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